The intersection form, logarithmic vector fields and the Severi strata in the discriminant of a plane curve singularity

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March 27, 2015

1 Introduction: the discriminant and its Severi strata

Two of the most basic invariants of a plane curve singularity $(C, 0)$ are its *Milnor number* μ and its *delta invariant* δ. If $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is a holomorphic map defining $(C, 0) = f^{-1}(0)$, then $\mu(C)$ is the dimension of the jacobian algebra $\mathcal{O}_{\mathbb{C}^2,0}\,/\,J_f$ and equals the dimension of the vanishing cohomology. If $n : \widetilde{C} \longrightarrow C$ denotes the normalisation of $(C, 0)$, then $\delta(C)$ is the dimension $n_*\,\mathcal{O}_{\widetilde{C}}\,/\,\mathcal{O}_C$ and equals the number of double points appearing in a generic perturbation of the map n. These invariants are related by the relation

$$\mu = 2\delta + r - 1$$

where r denotes the number of branches of $(C, 0)$. The number μ also appears as the number of parameters of an $\mathscr{R}_e$ miniversal deformation $F : (\mathbb{C}^2 \times \mathbb{C}^\mu, 0) \to (\mathbb{C}, 0)$ of the function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ defining $(C, 0)$. The restriction $\pi : X := F^{-1}(0) \to S = (\mathbb{C}^\mu, 0)$ is a versal deformation of the plane curve singularity $(C, 0)$. The fibre C_u over $u \in S$ is the curve defined by zero level of the deformed function $f_u := F(\ , u)$ and *discriminant* $D \subset S$ is the set of parameter values u for which the fibre C_u is singular. This set is stratified by the types of singularities that appear in the fibres. In this paper we will focus on the so-called *Severi strata*, where the sum of the delta-invariants add up to a value $\geq k$:

$$D(k) = \{u \in S : \sum_{p \in C_u} \delta(C_u, p) \geq k\}$$

Clearly $D(0) = S$ and $D(1) = D$, and as $D(i)$ is contained in $D(i-1)$ we obtain a chain

$$D(\delta) \subset D(\delta - 1) \subset \ldots \subset D(1) \subset D(0) = S$$

The smallest non-empty Severi stratum, $D(\delta)$, is the classical δ*-constant stratum*. The term "stratum" here is a bit of a misnomer, since the Severi strata are not in general smooth.

It is a classical fact that any curve singularity with $\delta = k$ can be deformed into a curve with precisely k A_1 points, a fact which explains the name *virtual number of double points* for the δ-invariant. Thus the set $D(kA_1)$ of parameter values u for which C_u has precisely k A_1 singularities is *dense* in $D(k)$. Moreover, $D(k)$ is smooth at such points, for there, by openness of versality, $D(k)$ is a normal crossing of k local irreducible components of the discriminant D. A curve singularity with δ-invariant $k > 1$ is also adjacent to a curve with one A_2 singularity and $k-1$ A_1 singularities. Hence $D(k)_{\mathrm{reg}} = D(kA_1)$.

Recently, these strata have been the subject of several papers and their geometry appears to hide some great mysteries. In [?] the multiplicity of $D(\delta)$ was shown to be equal to the Euler number of the compactified Jacobian of $(C, 0)$. This was further explored in [?], where multiplicities of the other $D(k)$ were related to the puntual Hilbert-schemes $Hilb^n(C, 0)$. Most surprisingly, these invariants turn out to be related to the HOMFLY-polynomial of the knot in the 3-sphere defined by $(C, 0)$, [?].

If the curve $(C, 0)$ is irreducible, its Milnor fibre C_u has just one boundary component, and it follows that the dual of the intersection form I_u on $H^1(C_u; \mathbb{C})$ is non-degenerate. In [?], Givental' and Varchenko used the period map associated to a non-degenerate 1-form η on the total space of the Milnor fibration of F, together with the Gauss-Manin connection, to pull back the intersection form from the cohomology bundle $\mathscr{H}^*$ over S to get a symplectic form Ω on $S \smallsetminus D$, and proved

Theorem 1.1. *(a) Ω extends to a symplectic form on S, and*

(b) the δ-constant stratum $D(\delta)$ in the discriminant is Lagrangian with respect to Ω.

Below we complement their results and show the following theorems.

Theorem 1.2. *All of the Severi strata are coisotropic with respect to Ω.*

The form Ω can also be used to obtain equations defining the Severi-strata. Let $\wedge^k\Omega$ be the k-fold wedge product of Ω. Although it is a regular form, it can be considered as an element of $\Omega^{2k}_S(\log D)$. Let I_k be the ideal generated by its coefficients with respect to a basis of $\Omega^{2k}_S(\log D)$.

Theorem 1.3. *For $k = 1, \ldots, \delta$, the Severi stratum $D(k)$ is defined by the ideal $I_{\delta-k+1}$:*

$$D(k) = V(I_{\delta-k+1}).$$

Equivalently, if $\chi_1, \ldots, \chi_\mu$ form a basis for the free module of logarithmic vector fields $\Theta_S(-\log D)$, then $D(k)$ is defined by the ideal generated by the Pfaffians of size $2\delta - 2k + 2$ of the skew matrix $(\Omega(\chi_i, \chi_j))_{1 \le i,j, \le \mu}$.

We do not know whether in general the ideals I_k are radical. Our calculations suggest that they are, but we have not been able to show this.

Givental' proved in [?] that for curve singularities of type A_{2k+1}, $D(\delta)$ is Cohen-Macaulay and it can be conjectured that this is always the case,[?]. In the first author's PhD thesis, [?], 1.3 was used to show that $D(\delta)$ is Cohen Macaulay also for E_6 and E_8; we give the argument below. Calculations using 1.3 suggest that the remaining Severi strata are Cohen-Macaulay in the case of A_{2k}, but show that for E_6 the stratum $D(2)$ is not Cohen-Macaulay .

In the process of proving these theorems we noticed that Ω determines a natural rank 2 maximal Cohen-Macaulay module over the discriminant D, which seems to be of independent interest.

2 Preliminaries

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define an isolated singularity $(C, 0)$ and let

$$1 = g_1, g_2, \ldots, g_\mu \in \mathcal{O}_{\mathbb{C}^{n+1},0}$$

be functions giving a basis for the jacobian algebra $\mathcal{O}/J_f$. We will consider be a good representative of a miniversal deformation of f of the form

$$F : B \times S \to \mathbb{C}, \quad F(x,u) = f(x) + \sum_{i=1}^{\mu} u_i g_i(x) \ ,$$

where B is a Milnor ball for C and S is a sufficiently small ball in $\mathbb{C}^\mu$,[?]. The set $X := F^{-1}(0)$ comes with a map $\pi : X \longrightarrow S$, with C_u as fibre over $u \in S$.

2.1 The critical space Σ

The relative critical set Σ of F is defined to be

$$\Sigma = \left\{(x,u) \in B \times S : \frac{\partial F}{\partial x_i}(x,u) = 0, i = 0, \ldots, n\right\}.$$

It is smooth and the projection $\pi : \Sigma \to S$ is a μ-fold branched cover: its fibre over $u \in S$ consists of the critical points of $F(-,u)$. As the partial derivatives form a regular sequence

$$\mathcal{O}_\Sigma = \mathcal{O}_{B\times S}/(\partial F/\partial x_0, \ldots, \partial F/\partial x_n)$$

is a free $\mathcal{O}_S$-module of rank μ. Miniversality of F is equivalent to the statement that the Kodaira-Spencer map

$$dF : \Theta_S \to \mathcal{O}_\Sigma, \quad \vartheta \mapsto \vartheta(F) = dF(\vartheta)$$

is an isomorphism. The set $X \cap \Sigma$ is the union over $u \in S$ of the set of singular points of C_u, and its image under π is the discriminant, D. For brevity we denote $X \cap \Sigma$ by $\widetilde{D}$. It is indeed the normalisation of D.

2.2 D as a free divisor

Let $\bar{F} : (B \times S, (0,0)) \to (\mathbb{C} \times S, (0,0))$ be the unfolding of f corresponding to the deformation F. Then $\Sigma \subset B \times \mathbb{C}^\mu$ is the (absolute) critical locus of $\bar{F}$. We write $\Delta = \bar{F}(\Sigma) \subset \mathbb{C} \times S$ for the set of critical values of $\bar{F}$. It is well known that Σ is the normalisation of Δ: it is smooth, and the map $\bar{F}| : \Sigma \to \Delta$ is generically one-to-one. Then $D = \Delta \cap \{0\} \times S$. As usual, $\Theta_{\mathbb{C}\times S}(-\log \Delta)$ denotes the $\mathcal{O}_{\mathbb{C}\times S}$-module of vector fields on $\mathbb{C} \times S$ which are tangent to Δ, and $\Theta_S(-\log D)$ denotes the $\mathcal{O}_S$-module of vector fields on S which are tangent to D.

Proposition 2.1. *(i) $\Theta_{\mathbb{C}\times S}(-\log \Delta)$ is the $\mathcal{O}_{\mathbb{C}\times S}$-module of vector fields on $\mathbb{C} \times S$ which are $\bar{F}$-liftable to $B \times S$.*
(ii) $\Theta_S(-\log D)$ is the $\mathcal{O}_S$-module of vector fields on S which are π-liftable to $V(F)$.

Proof. ([?]) (i) Let $\vartheta \in \Theta_{\mathbb{C}\times S}(-\log \Delta)$. Since $F| : \Sigma \to \Delta$ is the normalisation of Δ, there is a vector field $\tilde{\vartheta}_0$ on Σ lifting ϑ. For any extension $\tilde{\vartheta}_1$ of $\tilde{\vartheta}_0$ to $B \times S$, $\omega F(\vartheta) - tF(\tilde{\vartheta}_1)$ vanishes on Σ, and since the jacobian ideal $(\partial F/\partial x_1, \ldots, \partial F/\partial x_{n+1})$ is radical, there exists a second vector field ξ on $B \times S$ such that $\omega F(\tilde{\vartheta}_1) - tF(\tilde{\vartheta}_1 = tF(\xi)$. Then $\tilde{\vartheta}_1 + \xi$ is an $\bar{F}$-lift of ϑ.

Conversely, suppose $\tilde{\vartheta}$ is a $\bar{F}$-lift of ϑ. Then $\tilde{\vartheta}$ must be tangent to Σ, for the integral flows $\tilde{\Phi}_t$ and Φ_t of $\tilde{\vartheta}$ and ϑ satisfy $\Phi_1 \circ \bar{F} = \bar{F} \circ \Phi_t$, showing that $\tilde{\Phi}_t$ defines an isomorphism $\bar{F}^{-1}(u) \to \bar{F}^{-1}(\Phi_t(u))$, and must therefore map singular points of $\bar{F}^{-1}(u)$ to singular points of $\bar{F}^{-1}(\Phi_t(u))$. It follows that ϑ is tangent to Δ.

(ii) Let $i_0 : S \to \mathbb{C} \times S$ be the inclusion $u \mapsto (0, u)$. Then $D = i_u^{-1}(\Delta)$. Now i_0 is logarithmically transverse to Δ, i.e. transverse to the distribution $\Theta_{\mathbb{C}\times S}(-\log \Delta)$. If F is the standard deformation $f(x) + \sum_i u_i g_i$, with $g_\mu = 1$, then this transversality is obvious: the vector field $\partial/\partial t + \partial/\partial u_\mu$ on $\mathbb{C} \times S$ has $\bar{F}$-lift $\partial/\partial u_\mu$, and therefore lies in $\Theta_{\mathbb{C}\times S}(-\log \Delta)$. Any other miniversal deformation is parametrised $\mathscr{R}$-equivalent to the standard deformation, so the transversality holds there too.

Identifying $\mathbb{C}^\mu$ with $\{0\} \times \mathbb{C}^\mu$, from the logarithmic transversality of i_0 to Δ it follows that $\Theta_S(-\log D)$ is equal to the restriction to $\mathbb{C}^\mu$ of $\theta_{\mathbb{C}\times S}(-\log \Delta) \bigcap \theta_{\mathbb{C}\times S}(-\log(\{0\} \times S))$, and that every vector field in $\Theta_S(-\log D)$ extends to a vector field in $\Theta_{\mathbb{C}\times S}(-\log \Delta)$. Clearly, any lift to $\mathbb{C}^{n+1} \times S$ of a vector field in $\theta_{\mathbb{C}\times S}(-\log \Delta) \bigcap \theta_{\mathbb{C}\times S}(-\log(\{0\} \times S))$ must be tangent to $V(F)$, and any vector field whose $\bar{F}$-lift is tangent to $V(F)$ must lie in $\theta_{\mathbb{C}\times S}(-\log \Delta) \bigcap \theta_{\mathbb{C}\times S}(-\log(\{0\} \times S))$. $\square$

Therefore we have a diagram

$$\begin{tikzcd} & & \Theta_S \arrow[d, "dF"] & \Theta_S(-\log D) \arrow[l, hook'] \arrow[d, "\frac{dF}{F}"] & \\ 0 & \mathcal{O}_{\widetilde{D}} \arrow[l] & \mathcal{O}_\Sigma \arrow[l] & \mathcal{O}_\Sigma \arrow[l, "F"'] & 0 \arrow[l] \end{tikzcd} \tag{2.1}$$

where the vertical maps are isomorphisms. This diagram can be used to find a basis for $\Theta_S(-\log D)$. The germs $FdF(\partial/\partial u_i)$ generate $(F)\,\mathcal{O}_\Sigma$, therefore if

$$dF(\chi_i) = FdF\left(\frac{\partial}{\partial u_i}\right) \tag{2.2}$$

then the χ_i generate $\Theta_S(-\log D)$. This shows that $\Theta_S(-\log D)$ is a locally free module, so that D is a free divisor.

2.3 Stratification of D

The discriminant D is stratified in various ways. Of special relevance to us are the local $\mathscr{R}$ and $\mathscr{K}$ strata.

Suppose as before that $F : B \times S \to \mathbb{C}$ is a good representative of a versal deformation of f, where $\mathscr{B}$ is open in $\mathbb{C}^{n+1}$ and S is open in $\mathbb{C}^\mu$. Write $f_u = F(\_, u)$, and suppose that $p_1, \ldots, p_k$ are the critical points of f_u lying on $f_u^{-1}(0)$. For each point p_i, the germ

$$F : (B \times S, (p_i, u)) \to (\mathbb{C}, 0)$$

is an $\mathscr{R}_e$-versal deformation of the germ of f_u at p_i, by openness of versality. Hence there is a germ of submersion h_i from (S, u) to the base of an $\mathscr{R}_e$-miniversal deformation

$$G_i : (B \times \mathbb{C}^{\mu_i}, (x_i, 0)) \to (\mathbb{C}, 0)$$

of this germ, such that the germ of deformation $F : (B \times S, (p_i, u)) \to (\mathbb{C}, 0)$ is isomorphic to $h_i^*(G_i)$. We set

$$\mathscr{R}_i(u) = h_i^{-1}(0).$$

This is independent of the choice of miniversal deformation G_i and submersion h_i, since any two choices can be related by a commutative diagram of spaces and maps. Again by openness of versality, the $\mathscr{R}_i(u)$, $i = 1, \ldots, k$ are in general position with respect to one another, and we set

$$\mathscr{R}(u) = \bigcap_{i=1}^{k} \mathscr{R}_i(u).$$

T his is the *$\mathscr{R}$ stratum through u.* It is smooth of dimension $\mu - \sum_i \mu(f_u, p_i)$.

If in the above definition we replace $F : B \times S \to \mathbb{C}$ by the projection $V(F) \to S$, and replace each G_i by a $\mathscr{K}_e$-miniversal deformation H_i of the hypersurface singularity (C_u, p_i), then we obtain the $\mathscr{K}$-strata $\mathscr{K}_i(u)$ and their intersection $\mathscr{K}(u)$, the *$\mathscr{K}$-stratum through* u, which is once again smooth, by openness of versality, and has dimension $\mu - \sum_i \tau(C_u, p_i)$. Since $\mathscr{R} \subset \mathscr{K}$, $\mathscr{R}(u) \subset \mathscr{K}(u)$.

If for example the fibre C_u has k A_1 singularities and no other singular points, then $\mathscr{R}(u) = \mathscr{K}(u)$ and its germ at u coincides with the germ at u of the set of points u' such that $C_{u'}$ has k A_1 points and no other singularities.

Definition 2.2. *The logarithmic tangent space $T_u^{\log D} S$ is the vector subspace of $T_u S$ spanned at u by the logarithmic vector fields.*

Proposition 2.3. *One has the equality of vector spaces*

$$T_u^{\log D} = T_u \mathscr{K}(u).$$

Proof. We have the exact sequence

$$0 \to \Theta_S(-\log D) \to \Theta_S \to \pi_*(\mathcal{O}_{\widetilde{D}}) \to 0$$

which gives

$$\frac{\Theta_S}{\Theta_S(-\log D)} \simeq \pi_*(\mathcal{O}_{\widetilde{D}})$$

and so

$$\frac{T_u \mathbb{C}^\mu}{T_u^{\log} D} \simeq \frac{\Theta_S}{\Theta_S(-\log D) + \mathfrak{m}_{S,u}\Theta_{S,u}} \simeq \bigoplus_i T^1_{\mathscr{K}_e}(f_u, x_i)$$

This means that to show

$$T_u^{\log} D = T_u \mathscr{K}(u)$$

we need show only one inclusion. If $\vartheta \in \Theta_S(-\log D)_u$ then it has a lift $\widetilde{\vartheta}$ tangent to $V(F)$. The integral flows of ϑ and $\tilde{\vartheta}$, φ_t on (S, u) and $\widetilde{\varphi}_t$ on $V(F)$, satisfy $\pi \circ \tilde{\varphi}_t = \varphi_t \circ \pi$. It follows that $\tilde{\varphi}_t$ maps C_u to $C_{\varphi_t(u)}$, and therefore for each singular point p_i in C_u, the curve germ $\{\varphi_t(u) : t \in (\mathbb{C}, 0)\}$ lies in the set $\mathscr{K}_i(u)$ defined above. Hence $\{\varphi_t(u) : t \in (\mathbb{C}, 0)\} \subset \bigcap_i \mathscr{K}_i(u) = \mathscr{K}(u)$, and $\vartheta(0) \in T_u \mathscr{K}(u)$. $\square$

2.4 Isomorphism $\mathcal{O}_\Sigma \to \Omega_F$

A choice of a nowhere-vanishing relative $(n+1)$-form $\omega \in \Omega^{n+1}_{B \times S/S}$ determines an isomorphism

$$\mathcal{O}_\Sigma \simeq \Omega^{n+1}_F, \ \ g \mapsto g\omega$$

where

$$\Omega_F^{n+1} := \Omega^{n+1}_{B\times S/S}/dF \wedge \Omega^n_{B\times S/S}.$$

Such an isomorphism leads to many additional structures. First of all, there is a canonical perfect pairing, the *residue pairing*,

$$\mathrm{Res} : \Omega_F^{n+1} \times \Omega_F^{n+1} \to \mathcal{O}_S,$$

from which one obtains a perfect pairing on $\mathcal{O}_\Sigma$.

$$\langle\, \_\,, \_\,\rangle : \mathcal{O}_\Sigma \times \mathcal{O}_\Sigma \to \mathcal{O}_S\,.$$

Furthermore, because Ω^1_S and $\Omega^1_S(\log D)$ are $\mathcal{O}_S$-dual to Θ_S and $\Theta_S(-\log D)$, such a choice of ω also determines isomorphisms

$$\alpha : \Omega^1_S \to \mathcal{O}_\Sigma \quad \text{ and } \quad \beta : \Omega^1_S(\log D) \to \mathcal{O}_\Sigma$$

via the formulas

$$\langle dF(\vartheta), \alpha(\omega)\rangle = \omega(\vartheta), \quad \text{and} \quad \langle \frac{dF}{F}(\vartheta), \beta(\omega)\rangle = \omega(\vartheta).$$

As a result $\Theta_S, \Theta_S(-\log D), \Omega^1_S$ and $\Omega^1_S(\log D)$ are all identified with $\mathcal{O}_\Sigma$ and hence with one another. For example we have the isomorphism $k^{-1}\circ\beta : \Omega^1_S(\log D) \to \Theta_S$, where $k : \Theta_S \to \mathcal{O}_\Sigma$ is the Kodaira-Spencer map dF.

Note that multiplication by F on $\mathcal{O}_\Sigma$ is self-adjoint:

$$\langle g, Fh\rangle = \langle Fg, h\rangle.$$

As a result, if $\check{g}_i$, $i = 1, \ldots, \mu$ denotes the $\mathcal{O}_S$ basis of $\mathcal{O}_\Sigma$ dual to the basis $g_i = \partial F/\partial u_i$, $i = 1, \ldots, \mu$, then replacing $FdF(\partial/\partial u_i)$ in (2.2) by $\check{g}_i$, we get generators $\chi_1, \ldots, \chi_\mu$ for $\Theta_S(-\log D)$ whose matrix of coefficients with respect to the $\partial/\partial u_j$ is the symmetric matrix with i, j entry $\chi_{ij} = \langle \check{g}_i, F\check{g}_j\rangle$.

In our calculations in section 7 we always used such a basis. We note that if $\omega_1, \ldots, \omega_\mu$ is the basis for $\Omega^1(\log D)$ dual to $\chi_1, \ldots, \chi_\mu$ then

$$k^{-1}\beta(\omega_i) = \frac{\partial}{\partial u_i}, \quad \text{and} \quad k^{-1}\alpha(du_i) = \chi_i.$$

3 The Gauß-Manin connection

The study of the Gauß-Manin connection for hypersurface singularities was initiated by Brieskorn in [?] and has since then developed into a very detailed theory. We can only outline the parts of the theory that are relevant to our application. For a more detailled accounts we refer to [?], [?], [?], [?] and the original papers quoted there.

3.1 The cohomology bundle and its extensions

The spaces $H^n(X_u) = H^n(X_u; \mathbb{C})$ fit together into the cohomology bundle

$$H^* = \bigcup_{u\in S\setminus D} H^n(X_u)$$

over $S \setminus D$. It is a flat vector bundle and the associated sheaf of holomorphic sections

$$\mathscr{H}^* = H^* \otimes_C \mathcal{O}_{S\setminus D}$$

is equipped with a natural flat connection, the Gauss Manin connection,

$$\nabla : \mathscr{H}^* \to \mathscr{H}^* \otimes_{\mathcal{O}_S} \Omega^1_{S\setminus D} \tag{3.1}$$

As ususal, we write

$$\nabla_\vartheta : \mathscr{H}^* \longrightarrow \mathscr{H}^*$$

for the action of a vector field $\theta \in \Theta_{S\setminus D}$. The sheaf $\mathscr{H}^*$ over $S \setminus D$ has various extensions to S. Most relevant to us is the parameterised version of Brieskorn's module H':

$$\mathscr{H}' := \pi_*(\Omega^n_{X/S})/d\pi_*(\Omega^{n-1}_{X/S}). \tag{3.2}$$

A section of $\mathscr{H}$ over $U \subset S$ is represented by a (relative) holomorphic n-form η on $\pi^*(U) \subset X$. If $U \subset S \setminus D$ and $u \in U$, the restriction of η to the smooth fibre X_u is a closed form n-form and thus determines a cohomology class

$$[\eta|_{X_u}] \in H^n(X_u)$$

In this way one obtains an isomorphism $\mathscr{H}'(U) \to \mathscr{H}^*(U)$ and thus $\mathscr{H}'$ can be considered as an extension of $\mathscr{H}^*$, that is, there is a map of $\mathcal{O}_S$-modules

$$\mathscr{H}' \longrightarrow j_*\mathscr{H}^*,$$

which is an isomorphism over $S \setminus D$. ($j : S \setminus D \hookrightarrow S$ is the inclusion.) The sheaf $\mathscr{H}'$ is a locally free sheaf of rank μ, but for a general $\vartheta \in \Theta_S$ the Gauß-Manin connection map $\mathscr{H}'$ into a bigger extension $\mathscr{H}'' \supset \mathscr{H}'$. This second Brieskorn module $\mathscr{H}''$ can be defined as

$$\mathscr{H}'' := \pi_*\omega_{X/S}/d\pi_*(d\Omega^{n-1}_{X/S})$$

where $\omega_{X/S}$ denoted the relative dualising module, [**?**]. Elements from $\omega_{X/S}$ are most conviently described as residues of $n+1$-forms, that is, as Gelfand-Leray forms. There is an exact sequence

$$0 \longrightarrow \mathscr{H}' \longrightarrow \mathscr{H}'' \longrightarrow \Omega^{n+1}_{X/S} \longrightarrow 0 \tag{3.3}$$

When we restrict to logarithmic vector fields, the connection maps $\mathscr{H}'$ and $\mathscr{H}''$ to itself, so we have logarithmic connections

$$\nabla : \mathscr{H}' \longrightarrow \mathscr{H}' \otimes_{\mathcal{O}_S} \Omega^1_S(\log D)$$

$$\nabla : \mathscr{H}'' \longrightarrow \mathscr{H}'' \otimes_{\mathcal{O}_S} \Omega^1_S(\log D)$$

extending the Gauss-Manin connection (3.1). (As there is no possibility of confusion, we denote all these maps by the same name ∇)

The action of $\chi \in \Theta_S(-\log D)$ on a local section $[\eta]$ represented by a relative n-form η is given by the Lie-derivative with respect to a lift $\widetilde{\chi}$ of χ:

$$\nabla_\chi \eta = [Lie_{\widetilde{\chi}}(\eta)]$$

3.2 $\mathscr{H}'$ and the cohomology of singular fibres

We have seen that for $u \in S \setminus D$, the restriction of a global relative n-form η to a smooth fibre X_u determines a cohomology class

$$[\eta|_{X_u}] \in H^n(X_u)$$

If $u \in D$ then the fiber X_u is singular, but the form η still can be integrated over n-cycles in X_u and gives rise to a well defined cohomology class in $H^n(X_u)$. We sketch the argument. Suppose γ_1 and γ_2 are n-cycles in C_u and Γ is a $n+1$-chain in X_u with $\partial\Gamma = \gamma_1 - \gamma_2$. We can write $\Gamma = \Gamma' + \Gamma''$ where Γ' is a $n+1$-chain in the smooth part of C_u and $\Gamma'' = \Gamma \cap \bigcup_i B_\varepsilon(p_i)$, where the p_i are the singular points of C_u. Then

$$\int_{\gamma_1} \eta - \int_{\gamma_2} \eta = \int_{\partial\Gamma'} \eta + \int_{\partial\Gamma''} \eta.$$

The first integral on the right hand side vanishes by Stokes's Theorem. The contribution $\int_{\partial\Gamma''} \eta$ tends to 0 as $\varepsilon \to 0$, as the integrand is regular and one is integrating over ever smaller sets.

In general, if Z is any analytic space with singularities we can look at the de Rham-complex $(\Omega^\bullet_Z, d)$ of Kähler forms, and integration over p-cycles is well-defined and determines a *de Rham-evaluation map*

$$DR : H^p(\Gamma(Z, \Omega^\bullet_Z)) \to H^p(Z, \mathbb{C})$$

If Z is a Stein space, then this map is even *surjective*. The reason is the following: because Z is Stein, the group at the left hand side is equal to the p-th hypercohomology group $\mathbb{H}^p$ of the deRham-complex $(\Omega^\bullet_Z, d)$. The map of complexes $\mathbb{C}_Z \to (\Omega^\bullet_Z, d)$ (induced by the inclusion map $\mathbb{C}_Z \to \mathcal{O}_Z$) induces a map

$$\alpha : H^p(Z, \mathbb{C}) = \mathbb{H}^p(\mathbb{C}_Z) \to \mathbb{H}^p((\Omega^\bullet_Z, d)) = H^p(\Gamma(Z, \Omega^\bullet_Z))$$

and it is shown in [?], p.141, that DR is a *section* of the map α, i.e. $DR \circ \alpha = Id$. In particular, DR is surjective.

Propostion (8.5) of [?] provides a relative version of this argument, that we specialise to our situation of $\pi : X \longrightarrow S$. For this we look at the (truncated) relative deRham complex

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \Omega^1_{X/S} \longrightarrow \ldots \longrightarrow \Omega^{n-1}_{X/S} \longrightarrow \Omega^n_{X/S}$$

The cohomology sheaves are $\pi^{-1}\mathcal{O}_S$ in degree 0 and in degree n, where it is

$$\mathcal{H}^n := \Omega^{n+1}_{X/S}/d\Omega_{X/S},$$

a sheaf supported on $\widetilde{D}$. The direct image $(\pi_*\Omega^\bullet_{X/S}, d)$ also has two non-vanishing cohomologies, namely in degree 0 and in degree n where it is $\mathscr{H}'$. The two hypercohomology spectral sequences now produces a short exact sequence

$$0 \longrightarrow R^n\pi_*(\mathbb{C}_X) \otimes \mathcal{O}_S \xrightarrow{\alpha} \mathscr{H}' \xrightarrow{\beta} \pi_*\mathcal{H}^n \longrightarrow 0$$

Restriction to a (geometrical) fibre over u gives an exact sequence

$$0 \longrightarrow H^n(X_u) \longrightarrow \mathscr{H}'_u \longrightarrow \pi_*\mathcal{H}_u \longrightarrow 0$$

In the middle we have a vector space of dimension μ, at the right hand side a direct sum of vector spaces of dimension μ_i, the Milnor numbers of the singularties appearing in the fibre over u. So indeed

$$\dim H^n(X_u) = \mu - \sum \mu_i$$

The composition

$$R^n\pi_*(\mathbb{C}_X) \longrightarrow R^n\pi_*(\mathbb{C}_X) \otimes \mathcal{O}_S \xrightarrow{\alpha} \mathscr{H}'$$

is for any $u \in S$ a *section* to the deRham-evaluation map

$$DR_u : \mathscr{H}'_s \longrightarrow H^n(X_u, \mathbb{C})$$

Corollary 3.1. *For all $u \in S$, the deRham evaluation map*

$$\mathscr{H}'_u \to H^n(X_u); \eta \to [\eta|_{C_u}]$$

is surjective.

3.3 The period map

The theory of the period map was developed independently by Varchenko and K. Saito around the same time and has numerous applications. The basic idea is quite simple. Let us first fix a relative n-form η and a point $u \in S \setminus D$ and a horizontal basis $\gamma_1(s), \gamma_2(s), \ldots, \gamma_\mu(s) \in H^n(X_s)$ for points s in a neighbourhood U of u. The *period map*

$$P_\eta : U \longrightarrow \mathbb{C}^\mu, \quad s \mapsto (\int_{\gamma_1(s)} \eta, \int_{\gamma_2(s)} \eta, \ldots, \int_{\gamma_\mu(s)} \eta)$$

send a point s to the tuple of periods of the form η. By further parallel transport one obtains a (multi-valued) map

$$P_\eta : S \setminus D \longrightarrow \mathbb{C}^\mu$$

between spaces of the same dimension μ. The form η is called *non-degenerate* if it is non-degenerate at all points $u \in S \setminus D$, which means that P_η is a local isomorphism near u. Of course, this can be tested by looking at the derivative of this map, which can be identified with the map

$$\nabla P_{\eta,u} : T_u S \to H^1(X_u), \ \ \vartheta \mapsto [\nabla_\vartheta \eta | X_u] \in H^n(X_u)$$

which is the geometrical fibre at u of the sheaf map

$$\Theta_{S \setminus D} \longrightarrow \mathscr{H}^*, \quad \vartheta \mapsto \nabla_\vartheta \eta$$

This map extends to a sheaf map

$$\Theta_S \longrightarrow \mathscr{H}'', \quad \vartheta \mapsto \nabla_\vartheta \eta$$

which is an *isomorphism* in case η is non-degenerate.

Proposition 3.2. *A non-degenerate realtive n-form η gives rise to a commutative diagram*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathscr{H}' & \longrightarrow & \mathscr{H}'' & \longrightarrow & \Omega^{n+1}_{X/S} & \longrightarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \longrightarrow & \Theta_S(-\log D) & \longrightarrow & \Theta_S & \longrightarrow & \mathcal{O}_{\widetilde{D}} & \longrightarrow 0 \end{array}$$

(DAVID, COULD YOU REDO THIS DIAGRAM IN xypic*?) with exact rows and where the vertical maps are isomorphisms and where the map at the right hand side is induced by multiplication by $\omega = d\eta$.*

From this we get immediatly the following

Theorem 3.3. *If η is non-degenerate, then for each point $u \in S$ one obtains an isomorphism*

$$\nabla P_{\eta,u} : T_u^{logD} S \longrightarrow \mathscr{H}'_u$$

The composition with the deRham-evaluation map gives a surjection

$$DR \circ \nabla P_{\eta,u} : T_u^{logD} S \longrightarrow H^n(X_u)$$

Its restriction to

$$T_u\mathscr{R}(u) \subset T_u\mathscr{K}(u) = T_u^{logD} S \longrightarrow H^n(X_u)$$

is an isomorphism.

The last statement follows HERE WE STILL NEED AN ARGUMENT and the fact that both sides have the same dimension equal to $\mu(X) - \sum_i \mu_i$.

This statement was shown by Varchenko to hold in special cases and conjectured to hold in general, [**?**]. A proof, more or less along the above lines, was sketched to us by Hertling, [**?**].

4 The case of curves

We specialise to the case $n = 1$, so $C := X_0$ is a plane curve singularity. If C has r branches then by the formula of Milnor

$$\mu = 2\delta - r + 1,$$

and for $u \in S \setminus D$ the fibre $C_u := X_u$ is a smooth Riemann surface of genus $\delta - r + 1$ with r boundary circles. In the case where C is irreducible, then $\mu = 2\delta$ and for $u \notin D$, C_u is a smooth Riemann surface of genus δ. For $u \in D$ the curve C_u is a singular, say with singularities (C_u, p_i), $i = 1, 2, \ldots, N$ then its normalisation $\widetilde{C}_u$ has genus

$$\delta(C) - \sum_{i=1}^{N} \delta(C_u, p_i)$$

4.1 Intersection form

For fixed $u \in S$ let $C_u^* = C_u/\partial C_u$ be the closed Riemann surface obtained by shrinking ∂C_u to a point, and let $\tilde{C}_u$ and $\tilde{C}_u^*$ be the normalisations of C_u and C_u^*.

The diagram

$$\begin{tikzcd} \widetilde{C}_u \arrow[r, hook] \arrow[d, "n"] & \widetilde{C}_u^* \arrow[d, "n"] \\ C_u \arrow[r, hook, "j"] & C_u^* \end{tikzcd} \quad \text{gives rise to the diagram} \quad \begin{tikzcd} H^1(\widetilde{C}_u) & H^1(\widetilde{C}_u^*) \arrow[l, "\simeq"'] \\ H^1(C_u) \arrow[u, "n^*"] & H^1(C_u^*) \arrow[l, "\simeq"'] \arrow[u, "n^*"] \end{tikzcd}$$

in which the vertical arrows are surjections. Write I_u and $\tilde{I}_u$ for the intersection forms on C_u and $\widetilde{C}_u$. These are pulled back from the intersection forms on the closed curves C_u^* and $\widetilde{C}_u^*$ by means of the isomorphisms in the preceding diagram. Because $n_* : H_2(\widetilde{C}_u, \partial\widetilde{C}_u) \simeq H_2(C, \partial C)$, it follows by functoriality that

$$\widetilde{I}_u(n^*a, n^*b) = I_u(a, b), \tag{4.1}$$

Note that the form $\widetilde{I}_u$ is non-degenerate.

4.2 de Rham version of I_u

The pairing I_u has the following deRham description. We choose a a pair of collars $U \subset V \subset C_u$ around the boundary ∂C_u and a C^∞ bumb-function, equal to 1 on U and 0 outside V. If η is a holomorphic (Kähler) 1-form on C_u, it follows from the residue theorem that

$$\int_{\partial C} \eta = 0$$

By integration we can find a holomorphic function α on V with $d\alpha = \eta$ on V. The form η is cohomologuous to $\widetilde{\eta} := \eta - d\rho\alpha$ and as $\rho = 1$ on U and there $d\alpha = \eta$, it follows that $\widetilde{\eta}$ is a form with compact support, contained in $C \setminus U$. It is holomorphic and equal to η outside V, but only C^∞ on the annulus $V \setminus U$. One then has, using Stokes theorem

$$I_u([\eta], [\eta']) = I_u([\widetilde{\eta}], [\eta']) = -\int_{\partial C} \alpha\eta'$$

4.3 Extension to $\mathscr{H}^*$ and $\mathscr{H}'$

The pairings I_u on $H^1(C_u)$ combine to give a perfect duality

$$I : \mathscr{H}^* \times \mathscr{H}^* \to \mathcal{O}_S$$

over $S \setminus D$. Because of its topological origin, the intersection form is *horizontal* with respect to the Gauss-Manin connection: for any two sections s_1, s_2 of $\mathcal{H}^*$,

$$d\big(I(s_1, s_2)\big) = I(\nabla s_1, s_2) + I(s_1, \nabla s_2).$$

Using a relative version of the above deRham-description of the intersection pairing one obtains an extension of I, still called I, to $\mathscr{H}'$:

$$I : \mathscr{H}' \times \mathscr{H}' \to \mathcal{O}_S$$

For two sections η_1, η_2 of $\mathscr{H}'$ one has

$$I(\eta_1,\eta_2)(u) = I_u([\eta_1|C_u],[\eta_2|C_u])$$

4.4 Pulling back the intersection form

Using the period map one can pull-back the intersection form on $H^1(C_u)$ to obtain a 2-form on S. Let us first start with an arbitrary section $s \in \mathscr{H}^*$ over $S \setminus D$. From is we obtain a 2-form

$$\Omega = s^*I \in \Omega^2_{S \setminus D}$$

on $S \setminus D$ by the formula

$$\Omega(\theta_1,\theta_2) := I(\nabla_{\theta_1}s, \nabla_{\theta_2}s)$$

Proposition 4.1. *The form Ω is closed.*

Proof. This is 'clear' as we are pulling back the 'constant form I', but here is a nice direct calculation: if a, b and c are germs of commuting vector fields on S then

$$\begin{aligned} d(s^*I)(a,b,c) &= d\big(I(a,b)\big)(c) - d\big(I(a,c)\big)(b) + d(\big(I(b,c)\big)(a) \\ &= I(\nabla_c\nabla_a s, \nabla_b s) - I(\nabla_a s, \nabla_c \nabla_b s) \\ &\quad -I(\nabla_b\nabla_a s, \nabla_c s) + I(\nabla_a s, \nabla_b\nabla_c s) \\ &\quad +I(\nabla_a\nabla_b s, \nabla_c s) + I(\nabla_b s, \nabla_a\nabla_c s) \end{aligned} \tag{4.2}$$

Because a and b commute and ∇ is flat, $\nabla_a\nabla_b = \nabla_b\nabla_a$, and similarly for $\nabla_a\nabla_c$ and $\nabla_b\nabla_c$. This means that all terms on the right hand side in (4.2) cancel, except the first and last. These cancel because of the anti-symmetry of I. □

Theorem 4.2. ([?]) *If $s = \eta$ is a non-degenerate section of $\mathscr{H}'$, then Ω is itself non-degenerate and hence symplectic, and moreover extends to all of S as a symplectic form.*

5 Results

In [?] one find the formulation of a *principle* that the types of degeneration that occur in the fibres C_u are reflected in the lagrangian properties of the corresponding strata. Our results can be seen as a vindication of this principle in some special cases.

As before, we will consider the versal deformation $\pi : X \longrightarrow S$ of an irreducible curve singularity, a non-degenerate section η of the Brieskorn-module $\mathscr{H}'$ and the resulting symplectic form Ω on S, obtained by pulling back the intersection form on the fibres $H^1(C_u)$.

5.1 The rank of Ω on the logarithmic tangent space

Recall that for a point $u \in S$, the logarithmic tangent space $T_u^{logD}S \subset T_uS$ is the sub-space spanned by the logarithmic vector fields at u.

Theorem 5.1. *The rank of Ω restricted to $T_u^{logD}S$ is equal to the rank of I_u on $H^1(C_u)$, which is equal to dim $H^1(\widetilde{C}_u) = 2\delta(C) - \delta(C_u)$.*

Proof. Let $\mathscr{R}(u)$ and $\mathscr{K}(u)$ denote, respectively, the right-equivalence stratum and the $\mathscr{K}$-equivalence stratum containing u. Recall that by 3.3 the period map maps the space $T_u\mathscr{K}(u)$ surjectively to $H^1(C_u)$; its restriction to $T_u\mathscr{R}(u) \subseteq T_u\mathscr{K}(u)$ maps isomorphically to $H^1(C_u)$. From 4.3 it follows that the rank of Ω on $T_u^{\log}D$ at u is equal to the rank of the intersection form I_u on $H^1(C_u)$, which is equal to the rank of $H^1(\widetilde{C}_u)$, and therefore is equal to $\mu(C) - 2\delta(C_u) = 2\delta(C) - 2\delta(C_u)$. □

5.2 Coisotropicity of the Severi strata

Recall that a subspace V of a symplectic vector space W is *coisotropic* if $V^\perp \subset V$, where $V^\perp = \{w \in W : \langle v, w\rangle = 0 \text{ for all } v \in V\}$. A submanifold X of a symplectic manifold M is coisotropic if for all $x \in X$, T_xX is a coisotropic subspace of T_xM. A singular subset X of the symplectic manifold M is coisotropic if X_{reg} is coisotropic.

Theorem 5.2. *All the Severi strata*

$$D(\delta) \subset D(\delta - 1) \subset \cdots \subset D(1) = D$$

are coisotropic with respect to Ω.

Proof. Suppose that u is a regular point of $D(k)$, so C_u has exactly k ordinary double points as singularities. As $\mathscr{R}(u) = \mathscr{K}(u) = D(k)$ near u, the tangent space $T_uD(k)$ is the same as $T_u^{logD}S$. From theorem 5.1 the rank of $\Omega_{|T_uD(k)}$ is equal to $\mu - 2k$, hence dim $Ker\Omega_{|T_uD(k)} = k$. But from the non-degeneracy of Ω it follows that $T_uD(k)^\perp$ has dimension equal to the codimension of $D(k)$, namely k. Thus both sides in the relation

$$T_uD(k)^\perp \supset \ker(\Omega_u|_{T_uD(k)})$$

have dimension k, and are therefore equal. It follows that $T_uD(k)^\perp \subset T_uD(k)$. That is, $D(k)$ is coisotropic. □

The principle mentioned above explains this result by simply saying the near a regular point $u \in D(k)$ there are k cycles vanishing at u, which make up an isotropic subspace of in H_1. However, making this into a honest proof is another matter and leads to all the considerations outlined above.

5.3 Equations for the $D(k)$

Let $\chi_1, \ldots, \chi_\mu$ be a basis for for $\Theta_S(-\log D)$, and let $\omega_1, \ldots, \omega_\mu$ be the dual basis for $\Omega^1_S(\log D)$. Considering Ω as an element of $\Omega^2_S(\log D)$, it can be expressed as the sum

$$\Omega = \sum_{i<j} \Omega(\chi_i, \chi_j)\omega_i \wedge \omega_j.$$

We denote the skew matrix with i,j'th entry $\Omega(\chi_i, \chi_j)$ by $[\Omega]$. Then

$$\wedge^k\Omega = \sum_{1\leq i_1<\cdots<i_{2k}\leq\mu} \mathrm{Pf}([\Omega](i_1, \ldots, i_{2k}))\omega_{i_1} \wedge \cdots \wedge \omega_{i_{2k}}. \tag{5.1}$$

where $[\Omega](i_1, \ldots, i_{2k})$ is the submatrix of $[\Omega]$ consisting of rows and columns $i_1, \ldots, i_{2k}$ and Pf denotes its Pfaffian. The ideal generated by the coefficients of $\wedge^k\Omega$ with respect to the basis $\omega_{i_1} \wedge \cdots \wedge \omega_{i_{2k}}$ of $\Omega^{2k}(\log D)$ is the same as the ideal $\mathrm{Pf}_{2k}[\Omega]$ of $2k \times 2k$ Pfaffians of $[\Omega]$.

Theorem 5.3. *$D(k) = V(Pf_{2(\delta-k+1)}[\Omega])$. In particular, the δ-constant stratum $D(\delta)$ is defined by the entries of $[\Omega]$.*

Proof. Consider an aritrary $u \in S$. The rank of the matrix $[\Omega]$ at u is the rank of Ω restricted to the space of evaluations at u of the vector fields in $\Theta_S(-\log D)_u$, which is precisely the logarithmic tangent space $T_u^{logD}S$. Theorem 5.1 states that the rank of Ω on $T_u^{\log}D$ at u is equal to $2\delta(C) - 2\delta(C_u)$. As the rank of a skew-symmetric matrix is always even and equal to the size of the largest non-vanishing Pfaffian, it follows that $D(k)$ is precisely cut out by the Pfaffians of size $2(\delta - k + 1)$ of the matrix $[\Omega]$, i.e. $D(k) = V(Pf_{2(\delta-k+1)}([\Omega])$. □

A symplectic form Ω on a manifold S gives rise to a Poisson bracket $\{\_, \_\}$ on the sheaf of functions on S, as follows: Ω determines an isomorphism $\Omega_S^1 \to \Theta_S$ sending a 1-form α to a vector field $\alpha^\flat$. Then for functions f, g,

$$\{f, g\} = \Omega((df)^\flat, (dg)^\flat).$$

The vector field $\chi_f := (df)^\flat$ is called the *hamiltionian vector field* associated to f. If $V \subset S$ is a sub-variety and $I(V) \subset \mathcal{O}_S$ the ideal of functions vanishing on V, then it is easy to show that for a regular point $x \in V$ one has

$$T_xV^\perp = \{\chi_f(x) : f \in I(V)_x\}. \tag{5.2}$$

The following is well-known fact:

Proposition 5.4. *$V \subset S$ is coisotropic if and only if the ideal $I(V)$ is Poisson-closed:*

$$\{I(V), I(V)\} \subset I(V).$$

For convenience of the reader we include a proof.

Proof. Let $x \in V$ be a regular point and $v, w \in T_xV^\perp$, $f, g \in I(V)$ two functions with $\chi_f(x) = v$, $\chi_g(x) = w$. Then

$$\Omega(v, w) = \Omega(\chi_f(x), \chi_g(y)) = \{f, g\}(x)$$

From this we see that $\{f, g\}$ vanishes at x if and only if $\Omega(v, w) = 0$, which means that $T_xV^\perp \subset (T_xV^\perp)^\perp = T_xV$, that is, V is coisotropic. □

Thus, for each of the Severi strata $D(k)$, the ideal $I(D(k))$ is involutive. But note that an ideal defining a coisotropic subvariety is not necessarily involutive; the proof only shows that this holds if the ideal is radical.

Conjecture 5.5. *For all $k = 1, 2, \ldots, \delta$ (a) $Pf_{2k}([\Omega])$ is involutive. (b) $Pf_{2k}([\Omega])$ is radical.*

So by the theorem 5.3 $(b) \implies (a)$, as vanishing ideals of coisotropic varieties are involutive.Nevertheless, involutivity of the ideals $\mathrm{Pf}_{2k}([\Omega])$ may hold even without being radical.

6 The symplectic form as Extension

The matrix $[\Omega]$ can be considered as an endormorphism of $\mathcal{O}_S^\mu$ and its cokernel M_Ω defines a rank 2 Cohen-Macaulay module on $\mathcal{O}_D$, that sits in an exact sequence

$$0 \longleftarrow \mathcal{O}_{\widetilde{D}} \longleftarrow M_\Omega \longleftarrow \mathcal{O}_{\widetilde{D}} \longleftarrow 0$$

In fact, we show that the module $M_{\widetilde{D}}$ has a coordinate independent meaning, depending only on the choice of ω used in the definition of the period map. As such it represents a special element in the $\Omega_{\widetilde{D}}$-module

$$\mathrm{Ext}^1_D(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}})$$

6.1 Cohen-Macaulay modules

Let M be a maximal Cohen-Macaulay module (MCM) on the hypersurface singularity D defined by the equation h in the smooth space S. As is well known, the length 1 free resolution

$$0 \longleftarrow M \longleftarrow \mathcal{O}_S^\mu \xleftarrow{A} \mathcal{O}_S^\mu \longleftarrow 0 \tag{6.1}$$

of M over $\mathcal{O}_S$ becomes a 2-periodic resolution over $\mathcal{O}_D$:

$$0 \longleftarrow M \longleftarrow \mathcal{O}_D^\mu \xleftarrow{A} \mathcal{O}_D^\mu \xleftarrow{B} \mathcal{O}_D^\mu \xleftarrow{A} \mathcal{O}_D^\mu \longleftarrow \cdots . \tag{6.2}$$

It is possible to lift B to a matrix over $\mathcal{O}_S$ such that $AB = BA = hI_\mu$. We call such a matrix B (with entries in $\mathcal{O}_S$) a *companion matrix* to A. Any pair (A, B) with this property is known as a *matrix factorisation* of h, and gives rise to a pair of MCM's over $\mathcal{O}_D$, namely $\mathrm{Coker}A$ and $\mathrm{Coker}B$. By multiplying the entries of A by a suitable unit, it is always possible to suppose that $\det A = h^r$ for some $r \in \mathbb{N}$; r is in fact the rank of M as $\mathcal{O}_D$-module.

6.2 A construction

Let Ω be an invertible skew-symmetric $\mu \times \mu$ matrix. Of course, this forces μ to be even, and we now assume this. The matrix $A^t\Omega A$ is also skew. If the rank of M is 1, then we claim that $A^t\Omega A$ also presents an MCM over $\mathcal{O}_D$. To see this, let $\mathrm{Pf}(W)$ denote the Pfaffian of a skew matrix W, and notice that $\mathrm{Pf}(A^t\Omega A) = \sqrt{\det(A^t\Omega A)} = \sqrt{(\det\ A)^2} = \pm h$ (the second equality only up to multiplication by a unit in $\mathcal{O}_S$). A $\mu \times \mu$ skew matrix W has a "skew-adjugate" matrix W^{skAdj} such that $WW^{\mathrm{skAdj}} = W^{\mathrm{skAdj}}W = \mathrm{Pf}(W)I_\mu$; thus in our case $(A^t\Omega A, (A^t\Omega A)^{\mathrm{skAdj}})$ is a matrix factorisation of h.

The existence of the regular companion matrix $(A^t\Omega A)^{\mathrm{skAdj}}$ makes possible its explicit determination: from

$$(A^t\Omega A)^{\mathrm{skAdj}} \cdot A^t\Omega A = hI_\mu$$

working in the field of fractions of $\mathcal{O}_S$ we deduce

$$(A^t\Omega A)^{\mathrm{skAdj}} = h \cdot (A^t\Omega A)^{-1} = h \cdot \frac{1}{h} A^{\mathrm{Adj}}\Omega^{-1}\frac{1}{h}(A^t)^{\mathrm{Adj}} = \frac{1}{h}A^{\mathrm{Adj}}\Omega^{-1}(A^t)^{\mathrm{Adj}}.$$

We denote by N_Ω the maximal Cohen-Macaulay $\mathcal{O}_D$-module $\mathrm{Coker}A^t\Omega A$. The case that concerns us is where A is a symmetric Saito matrix χ of the discriminant D, and Ω is the matrix of the symplectic form on the base S of the versal deformation, obtained by pulling back the intersection form from the cohomology bundle. In this case we call N_Ω the *intersection module.* The symmetry of χ adds significantly to the picture, as we now describe.

Assume that A is symmetric and consider the following commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_S^\mu \arrow[d, "A"'] & \mathcal{O}_S^\mu \arrow[l, "\Omega A"'] \arrow[d, "A\Omega A"] & \mathcal{O}_S^\mu \arrow[l, "="'] \arrow[d, "A"] & \\
 & \mathcal{O}_S^\mu \arrow[d, "q"'] & \mathcal{O}_S^\mu \arrow[l, "="] \arrow[d, "p"] & \mathcal{O}_S^\mu \arrow[l, "A\Omega"'] \arrow[d, "q"] & \\
0 & M \arrow[l] \arrow[d] & N_\Omega \arrow[l, "j"'] \arrow[d] & M \arrow[l, "i"'] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{6.3}
$$

where the columns are free resolutions.

Proposition 6.1. *The sequence*

$$
\begin{tikzcd}
0 & M \arrow[l] & N_\Omega \arrow[l, "j"'] & M \arrow[l, "i"'] & 0 \arrow[l]
\end{tikzcd}
\tag{6.4}
$$

is exact.

Proof. j is obviously surjective. i is injective because

$$iq(m) = 0 \implies A\Omega m = A\Omega A n \text{ for some } n \implies A\Omega(m - An) = 0 \implies m = An,$$

so $q(m) = 0$. The sequence is exact at N_Ω because

$$jpn = 0 \iff n = Am \text{ for some } m \iff n = A\Omega m' \text{ for some } m' \iff pn = iqm' \text{ for some } m'.$$

$\square$

Remark 6.2. *If the matrix A is not assumed symmetric, the argument of the proof of 6.1 produces an exact sequence*

$$
\begin{tikzcd}
0 & CokerA^t \arrow[l] & CokerA^t\Omega A \arrow[l] & M \arrow[l, "i"'] & 0 \arrow[l]
\end{tikzcd} .
\tag{6.5}
$$

To express the extension in more invariant terms, we augment diagram (6.3) to

$$
\begin{tikzcd}
 & \mathcal{O}_S^\mu \arrow[dl, "b\chi"'] \arrow[d, "\chi"] & \mathcal{O}_S^\mu \arrow[l, "\Omega\chi"'] \arrow[d, "\chi\Omega\chi"] & \mathcal{O}_S^\mu \arrow[l, "="'] \arrow[d, "\chi"'] \arrow[dr, "b\chi"] & \\
\Theta_S \arrow[dr, "dF"'] & \mathcal{O}_S^\mu \arrow[l, "\simeq"', "b"] \arrow[d] & \mathcal{O}_S^\mu \arrow[l, "="'] \arrow[d] & \mathcal{O}_S^\mu \arrow[l, "\chi\Omega"] \arrow[d] \arrow[r, "\simeq", "b"'] & \Theta_S \arrow[dl, "dF"] \\
0 & \mathcal{O}_{\widetilde{D}} \arrow[l] \arrow[d] & N_\Omega \arrow[l] \arrow[d] & \mathcal{O}_{\widetilde{D}} \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{6.6}
$$

Here b is the isomorphism sending the i'th basis element of $\mathcal{O}_S^\mu$ to $\partial/\partial u_i$. Condensing, this gives a less coordinate-dependent version of (6.3):

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_S^\mu \arrow[d, "b\chi"'] & \mathcal{O}_S^\mu \arrow[l, "\Omega\chi"'] \arrow[d, "\chi\Omega\chi"] & \mathcal{O}_S^\mu \arrow[l, "="'] \arrow[d, "b\chi"] & \\
 & \Theta_S \arrow[d, "dF"'] & \mathcal{O}_S^\mu \arrow[l, "b"] \arrow[d, "p"] & \Theta_S \arrow[l, "\chi\Omega b^{-1}"'] \arrow[d, "dF"] & \\
0 & \mathcal{O}_{\widetilde{D}} \arrow[l] \arrow[d] & N_\Omega \arrow[l, "j"'] \arrow[d] & \mathcal{O}_{\widetilde{D}} \arrow[l, "i"'] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{6.7}
$$

6.3 Invariance

We show that the extension in the bottom row of diagram (6.7) is independent of the choices involved in its construction, by describing the construction in a coordinate free way. We use the double duality isomorphism described in Subsection **??**,

$$\Omega^1(\log D) \xrightarrow{\Phi} \Theta_S$$

defined by

$$\langle dF(\Phi(\omega)), \frac{1}{F} dF(v)\rangle = \omega(v) \qquad \text{for all } v \in \Theta_S(-\log D).$$

We also use "contraction by Ω",

$$\mathscr{E} : \Theta_S \to \Omega_S^1$$

defined by $\mathscr{E}(\chi) = \iota_\chi(\Omega)$, and the inclusions

$$\Theta_S(-\log D) \xrightarrow{i} \Theta_S$$

and

$$\Omega_S^1 \xrightarrow{j} \Omega^1(\log D)\ .$$

Lemma 6.3. *We have the following equalities of matrices:*

$$\chi = [i]_\partial^B \qquad\qquad \chi^t\Omega\chi = [\Phi\,\mathscr{E}\,i]_\partial^B \qquad\qquad \chi = [\Phi j]_\partial^d$$

where B is the basis $\chi_1, \ldots, \chi_\mu$ of $\Theta_S(-\log D)$, ∂ is the basis $\partial/\partial u_1, \ldots, \partial/\partial u_\mu$ of Θ_S, and d is the basis $dx_1, \ldots, dx_\mu$ of Ω_S^1.

Proof. The first equality is the definition of the matrix χ. For the second, as observed above $\chi\Omega\chi = [\mathscr{E} i]_\omega^B$ where ω is the basis $\omega_1, \ldots, \omega_\mu$ of $\Omega^1(\log D)$. Since $[\Phi]_\partial^\omega$ is the identity matrix, the second equality follows. For the last equality, recall that by definition of Φ,

$$\langle dF(\Phi(du_i)), \frac{1}{F} dF(v)\rangle = du_i(v)$$

for all $v \in \Theta_S(-\log D)$. In particular this is true if $v = \chi_k$, so, since we have chosen χ_k so that $\frac{1}{F}dF(\chi_k)$ is the k'th member of the basis $\check{E}_1, \ldots, \check{E}_\mu$ of $\mathcal{O}_{\Sigma_F}$ dual to the basis $E := \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_\mu}$, we have

$$\langle dF(\Phi(du_i)), \check{E}_k\rangle = \langle dF(\Phi(du_i)), \frac{1}{F}dF(\chi_k)\rangle = du_i(\chi_k).$$

By definition of "dual basis", this means that

$$\begin{aligned} dF(\Phi(du_i)) &= \sum_k \langle dF(\Phi(du_i)), \check{E}_k\rangle E_k = \sum_k du_i(\chi_k)E_k \\ &= \sum_k du_i(\chi_k) dF\left(\frac{\partial}{\partial u_k}\right) \\ &= dF\left(\sum_k du_i(\chi_k)\frac{\partial}{\partial u_k}\right). \end{aligned}$$

But by symmetry of the matrix of coefficients χ, we have $du_i(\chi_k) = du_k(\chi_i)$, so

$$dF(\Phi(du_i)) = dF\left(\sum_k du_k(\chi_i)\frac{\partial}{\partial u_k}\right) = dF(\chi_i).$$

As $dF : \Theta_S \to \mathcal{O}_{\Sigma_F}$ is injective, this means that

$$\Phi(du_i) = \chi_i.$$

$\square$

Using these equalities, and omitting inclusions i and j to avoid clutter, diagram 6.7 becomes

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ & \Omega^1_S \arrow[d, "\Phi"'] & \Theta_S(-\log D) \arrow[l, "\mathscr{E}"'] \arrow[d, "\Phi\mathscr{E}"] & \Theta_S(-\log D) \arrow[l, "="'] \arrow[d, hook] & \\ & \Theta_S \arrow[d, "dF"] & \Theta_S \arrow[l, "="'] \arrow[d] & \Theta_S \arrow[l, "\Phi\mathscr{E}"'] \arrow[d, "dF"] & \\ 0 & \mathcal{O}_{\Sigma^0_F} \arrow[l] \arrow[d] & N_\Omega \arrow[l] \arrow[d] & \mathcal{O}_{\Sigma^0_F} \arrow[l] \arrow[d] & 0 \arrow[l] \\ & 0 & 0 & 0 & \end{tikzcd} \tag{6.8}$$

showing that provided the Gorenstein pairing on $\mathcal{O}_{\Sigma_F}$, and therefore Φ, are chosen canonically, the extension in the bottom row of the diagram depends only on F and on the symplectic form.

6.4 Calculation of Ext groups

As before, $D \subset \mathbb{C}^\mu$ is a free divisor with symmetric Saito matrix χ, and Gorenstein normalisation $\widetilde{D}$. We pick a symmetric presentation Λ for $\mathcal{O}_{\widetilde{D}}$ over $\mathcal{O}_D$, with respect to a generating set $g_1, \ldots, g_\mu$ for $\mathcal{O}_{\widetilde{D}}$. In the case where D is the discriminant, then we can take $\Lambda = \chi$. We denote by $\mathscr{C}$ the conductor ideal in $\mathcal{O}_D$ and in $\mathcal{O}_{\widetilde{D}}$. Over $\mathcal{O}_{\widetilde{D}}$, $\mathscr{C}$ is a principal ideal, since $\widetilde{D}$ is Gorenstein; it is generated by the submaximal minor m_1^1 of Λ, if $g_1 = 1$.

Lemma 6.4. *(i) Both $Ext^1_D(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}})$ and $Ext^2_D(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}})$ are $\mathcal{O}_{\widetilde{D}}/\mathscr{C}$-modules.*

(ii)
$$Ext^1_{\mathcal{O}_D}(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}}) \simeq \frac{\{\mathcal{O}_{\widetilde{D}}\text{-syzygies of } g_1, \ldots, g_\mu\}}{\mathcal{O}_{\widetilde{D}} \cdot \{\mathcal{O}_D\text{-syzygies of } g_1, \ldots, g_\mu\}};$$

(iii)
$$Ext^2_{\mathcal{O}_D}(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}}) \simeq \mathcal{O}_{\widetilde{D}}/\mathscr{C}.$$

Proof. Let $K_\bullet$ denote the free resolution

$$0 \longleftarrow \mathcal{O}_{\widetilde{D}} \xleftarrow{p} \mathcal{O}_D^\mu \xleftarrow{\Lambda} \mathcal{O}_D^\mu \xleftarrow{\Lambda^{\mathrm{Adj}}} \mathcal{O}_D^\mu \xleftarrow{\Lambda} \mathcal{O}_D^\mu \longleftarrow \cdots \tag{6.9}$$

Identifying $\mathrm{Hom}_{\mathcal{O}_D}(\mathcal{O}_D, \mathcal{O}_{\widetilde{D}})$ with $\mathcal{O}_{\widetilde{D}}$, $\mathrm{Ext}^j_{\mathcal{O}_D}(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}})$ is the j'th cohomology of the complex $\mathrm{Hom}_D(K_\bullet, \mathcal{O}_{\widetilde{D}})$ equal to

$$0 \longrightarrow \mathcal{O}_{\widetilde{D}}^\mu \xrightarrow{\Lambda} \mathcal{O}_{\widetilde{D}}^\mu \xrightarrow{\Lambda^{\mathrm{adj}}} \mathcal{O}_{\widetilde{D}}^\mu \xrightarrow{\Lambda} \cdots$$

where we use the fact that $\Lambda^t = \Lambda$ and $(\Lambda^{\mathrm{adj}})^t = \Lambda^{\mathrm{adj}}$. The modules and differentials in the complex are $\mathcal{O}_{\widetilde{D}}$-linear, so both Ext^1 and Ext^2 are $\mathcal{O}_{\widetilde{D}}$-modules.

Now assume that $g_1 = 1$. Let m^i_j be the signed (i,j) cofactor of Λ. The symmetry of Λ, and the fact that in $\mathcal{O}_{\widetilde{D}}$ $m^i_j g_k = m^k_j g_i$ for $1 \le i,j,k \le \mu$ (by Cramer's rule – see [**?**]), mean that $m^i_j = m^1_1 g_i g_j$, so the i,j entry in Λ^{adj}, as an element in $\mathcal{O}_{\widetilde{D}}$, is $m^1_1 g_i g_j$. As m^1_1 is not a zero-divisor on $\mathcal{O}_{\widetilde{D}}$,

$$\ker \Lambda^{\mathrm{adj}} = \{(a_1, \ldots, a_\mu)^t \in \mathcal{O}_{\widetilde{D}}^\mu : \sum_i a_i g_i = 0\}.$$

Meanwhile the image of Λ is the $\mathcal{O}_{\widetilde{D}}$-submodule of $\mathcal{O}_{\widetilde{D}}^\mu$ generated by the columns of Λ. This proves (ii).

For Ext^2, note that $\ker \Lambda$ contains the (free) $\mathcal{O}_{\widetilde{D}}$-submodule $\mathcal{O}_{\widetilde{D}}\,\mathbf{g}$ of $\mathcal{O}_{\widetilde{D}}^\mu$ generated by $\mathbf{g} := (g_1, \ldots, g_\mu)^t$. If $\mathbf{h} := (h_1 \ldots, h_\mu) \in \ker \Lambda$ then $\mathbf{h} - h_1\mathbf{g} \in \ker \Lambda$, and has 0 as first entry. This implies that the matrix Λ with first column deleted kills $\mathbf{h} - h_1\mathbf{g}$ with first row deleted. Since m^1_1 is not a zero divisor on $\mathcal{O}_{\widetilde{D}}$, we must have $\mathbf{h} - h_1\mathbf{g} = 0$. Thus $\ker \Lambda = \mathcal{O}_{\widetilde{D}}\,\mathbf{g}$. As $\mathrm{Im}(\Lambda^{\mathrm{adj}}) = m^1_1\mathbf{g}$, the statement for Ext^2 follows.

It is clear from (ii) and (iii) that both Ext^1 and Ext^2 are annihilated by $\mathscr{C}$. □

Proposition 6.5. *$Ext^1_D(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}})$ is a maximal Cohen-Macaulay module over $\mathcal{O}_{\widetilde{D}}/\mathscr{C}$ presented by the matrix $\widetilde{\Lambda} := \Lambda$ with first row and column deleted.*

Proof. Let $e_1,\ldots,e_\mu$ be standard generators of the free module $\mathcal{O}_D^\mu$ projecting to $g_1,\ldots,g_\mu$ in the presentation of $\mathcal{O}_{\widetilde{D}}$ over $\mathcal{O}_D$. As $\mathcal{O}_{\widetilde{D}}$-generators for the module of $\mathcal{O}_{\widetilde{D}}$-syzygies of $g_1,\ldots,g_\mu$ we can take $s_k := g_1e_k - g_ke_1$ for $k=2,\ldots,\mu$. For if $\sum_j \alpha_j g_j = 0$ is any $\mathcal{O}_{\widetilde{D}}$ relation among the g_j, then

$$\alpha_2 s_2 + \cdots + \alpha_\mu s_\mu = -\sum_{j=2}^{\mu}\alpha_j g_j e_1 + \sum_{j=2}^{\mu}\alpha_j e_j = \alpha_1 g_1 e_1 + \sum_{j=2}^{\mu}\alpha_j e_j = \sum_{j=1}^{\mu}\alpha_j e_j,$$

the last equality because $g_1 = 1$. Denote the class of s_k in $\mathrm{Ext}^1_D(\mathcal{O}_{\widetilde{D}},\mathcal{O}_{\widetilde{D}})$ by $\bar{s}_k$. Each $\mathcal{O}_D$-syzygy among the g_i

$$\lambda_j^1 e_1 + \cdots + \lambda_j^\mu e_\mu,$$

gives rise to the relation

$$\lambda_j^2 \bar{s}_2 + \cdots + \lambda_j^\mu \bar{s}_\mu = 0 \tag{6.10}$$

among the $\bar{s}_k$, since

$$\begin{aligned}\lambda_j^1 e_1 + \cdots + \lambda_j^\mu e_\mu &= \big(\lambda_j^1 + \lambda_j^2 g_2 + \cdots + \lambda_j^\mu g_\mu\big)e_1 + \lambda_j^2(e_2 - g_2e_1) + \cdots + \lambda_j^\mu(e_\mu - g_\mu e_1)\\ &= \lambda_j^2 s_2 + \cdots + \lambda_j^\mu s_\mu\end{aligned}$$

(for the last equality we have used the facts that $g_1 = 1$ and that $\sum_i \lambda_j^i g_i = 0$).

We claim that every $\mathcal{O}_{\widetilde{D}}$-relation among the $\bar{s}_k$ in $\mathrm{Ext}^1_D(\mathcal{O}_{\widetilde{D}},\mathcal{O}_{\widetilde{D}})$ is an $\mathcal{O}_{\widetilde{D}}$-linear combination of the relations (6.10), for $j=2,\ldots,\mu$. If

$$\alpha_2\bar{s}_2 + \cdots + \alpha_\mu\bar{s}_\mu = 0 \tag{6.11}$$

is any such relation, then by 6.5(ii), $\alpha_2 s_2 + \cdots + \alpha_\mu s_\mu$ is an $\mathcal{O}_{\widetilde{D}}$-linear combination of $\mathcal{O}_D$- syzygies of the g_i, and thus equal to $\sum_i \beta_i(\lambda_i^1,\ldots,\lambda_i^\mu)$ for some $\beta_1,\ldots,\beta_\mu \in \mathcal{O}_{\widetilde{D}}$. That is,

$$\begin{pmatrix} -g_2 & -g_3 & \cdots & \cdots & -g_\mu \\ 1 & 0 & \vdots & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}\begin{pmatrix}\alpha_2\\ \vdots \\ \vdots \\ \alpha_\mu\end{pmatrix} = \Lambda\begin{pmatrix}\beta_1 \\ \vdots \\ \vdots \\ \vdots \\ \beta_\mu\end{pmatrix} \tag{6.12}$$

Delete the first row of (6.12): we get

$$\begin{pmatrix}\alpha_2\\ \vdots \\ \alpha_\mu\end{pmatrix} = \beta_1\begin{pmatrix}\lambda_1^2\\ \vdots \\ \lambda_1^\mu\end{pmatrix} + \cdots + \beta_\mu\begin{pmatrix}\lambda_\mu^2\\ \vdots \\ \lambda_\mu^\mu\end{pmatrix} = (\beta_2 - \beta_1 g_2)\begin{pmatrix}\lambda_2^2\\ \vdots \\ \lambda_2^\mu\end{pmatrix} + \cdots + (\beta_\mu - \beta_1 g_\mu)\begin{pmatrix}\lambda_\mu^2\\ \vdots \\ \lambda_\mu^\mu\end{pmatrix} = \tilde{\Lambda}\begin{pmatrix}\beta_2 - \beta_1 g_2 \\ \vdots \\ \beta_\mu - \beta_1 g_\mu\end{pmatrix}.$$

The second equality here simply uses the fact that $\Lambda\mathbf{g} = 0$ to express the first column of Λ as a linear combination of the remaining columns.

We have shown that $\mathrm{Ext}^1_D(\mathcal{O}_{\widetilde{D}},\mathcal{O}_{\widetilde{D}})$ is presented by the matrix $\tilde{\Lambda}$. Since $(\det\widetilde{\Lambda})\,\mathcal{O}_{\widetilde{D}} = \mathscr{C}$, $\mathrm{Ext}^1_D(\mathcal{O}_{\widetilde{D}},\mathcal{O}_{\widetilde{D}}) = \mathrm{Coker}\,\widetilde{\Lambda}$ is a maximal Cohen-Macaulay $\mathcal{O}_{\widetilde{D}}/\mathscr{C}$-module. $\square$

In [?] it is shown that if $n : \widetilde{D} \to D$ has corank 1 then $\mathrm{Coker}\widetilde{\Lambda} \simeq \pi_* \mathcal{O}_{D^2(n)}$, where, by $D^2(n)$, we mean the double-point scheme of the map n:

$$D^2(n) = \text{closure}\{(x_1, x_2) \in \widetilde{D} \times \widetilde{D} : x_1 \neq x_2, n(x_1) = n(x_2)\}.$$

The isomorphism fails if n has corank > 1. The map $n : \widetilde{D} \to D$, normalising the discriminant in the base of a versal deformation, has corank 1 exactly for the A_μ singularities. Thus, for the A_μ, and only for these, $\mathrm{Ext}^1_D(\mathcal{O}_{\widetilde{D}}, \mathcal{O}_{\widetilde{D}}) \simeq \mathcal{O}_{D^2(n)}$.

7 Computations and Examples

It was described in [] how the symplectic form Ω can be computed in the case of irreducible quasi-homogeneous curve singularities. The projective closure of such a curve has a unique point at infinity ∞.

Proposition 7.1. *Let C be a curve, $\infty \in C$ a smooth point and ω, η two meromorphic differential form, holomorphic on $C \setminus \{\infty\}$. Then the intersection form of the cohomology classes $[\omega], [\eta] \in H^1(C)$ is*

$$I([\omega], [\eta]) = 2\pi i Res_\infty(\alpha\eta)$$

where α is a meromorphic function in a neighbourhood of ∞ with $d\alpha = \omega$.

Proof. Choose two small open discs $U \subset V \subset C$ around ∞, and a $\mathcal{C}^\infty$ bump function ρ on C, equal to 1 on U and 0 outside V. Choose an α meromorphic on V with $d\alpha = \omega$. Then $\omega - d(\rho\alpha$ is a C^∞ compactly supported form, cohomologous to $[\omega]$. Using $\omega \wedge \eta = 0$, we find

$$I([\omega], [\eta]) = -\int_C d(\rho\alpha)\eta = -\int_U d(\rho\alpha \cdot \eta)$$

and by Stokes theorem

$$-\int_U d(\rho\alpha \cdot \eta) = -\int_{\partial U} \alpha\eta$$

which, noticing the reverse of orientation in the boundary, gives the above formula. □

This proposition can be used to calculate intersections using Laurent-series exapansions. If the curve C is given by an affine equation $f(x, y) = 0$ and has a single point at infinity, we can find a Laurent parametrisation of C around ∞

$$x(t), y(t) \in \mathbb{C}[[t]][1/t]$$

If $\omega = A(x, y)dx$ and $\eta = B(x, y)dx$ are the differential forms on C, then by substitution we obtain expansions

$$\omega = a(t)dt, \eta = b(t)dt$$

where $a(t), b(t) \in \mathcal{C}[[t]][1/t]$ are Laurent series. Integrating up one we find

$$\alpha(t) = \int a(t)dt \in \mathcal{C}[[t]][1/t]$$

and can compute the cohomological intersection as:

$$I([\omega], [\eta]) = Res_0 \alpha(t) b(t) dt$$

Proposition 7.2. ([?]) *Suppose that f is quasihomogeneous. Then for $\omega = dx \wedge dy$, the period map P_ω is non-degenerate.*

Case A_4

We consider the versal deformation of A_4 given by

$$F(x,a,b,c,d) = x^5 + ax^3 + bx^2 + cx + d.$$

We take the symmetric basis for $\Theta_S(-\log D)$ with Saito matrix

$$\chi := \begin{pmatrix} 10a & 15b & 20c & 25d \\ 15b & -6a^2+20c & -4ab+25d & -2ac \\ 20c & -4ab+25d & -6b^2+10ac & -3bc+15ad \\ 25d & -2ac & -3bc+15ad & -4c^2+10bd \end{pmatrix} \tag{7.1}$$

The symplectic form pulled back by the period mapping induced by the 1-form ydx is

$$\Omega = a da \wedge db + da \wedge dd + 3db \wedge dc. \tag{7.2}$$

Therefore the ideal of entries of the matrix $\chi\Omega\chi$, defining the δ-constant stratum $D(2)$, is generated by

$$a^4 + \frac{27}{4}ab^2 - 9a^2c + 20c^2 - \frac{25}{2}ad, \quad a^3b + \frac{27}{4}b^3 - 9abc - 10a^2d + 50cd \tag{7.3}$$

and

$$a^3c + \frac{27}{4}b^2c - 4ac^2 - 20abd + \frac{125}{4}d^2$$

Case A_6

A versal deformation of A_6 is given by

$$F(x,a,b,c,d,e,f) = x^7 + ax^5 + bx^4 + cx^3 + dx^2 + ex.$$

We take the basis of $\Theta_S(-\log D)$ with Saito matrix

$$\begin{pmatrix} 2a & 3b & 4c & 5d & 6e & 7f \\ 3b & -\frac{10}{7}a^2+4c & -\frac{8}{7}ab+5d & -\frac{6}{7}ac+6e & -\frac{4}{7}ad+7f & -\frac{2}{7}ae \\ 4c & -\frac{8}{7}ab+5d & -\frac{12}{7}b^2+2ac+6e & -\frac{9}{7}bc+3ad+7f & -\frac{6}{7}bd+4ae & -\frac{3}{7}be+5af \\ 5d & -\frac{6}{7}ac+6e & -\frac{9}{7}bc+3ad+7f & -\frac{12}{7}c^2+2bd+4ae & -\frac{8}{7}cd+3be+5af & -\frac{4}{7}ce+4bf \\ 6e & -\frac{4}{7}ad+7f & -\frac{6}{7}bd+4ae & -\frac{8}{7}cd+3be+5af & -\frac{10}{7}d^2+2ce+4bf & -\frac{5}{7}de+3cf \\ 7f & -\frac{2}{7}ae & -\frac{3}{7}be+5af & -\frac{4}{7}ce+4bf & -\frac{5}{7}de+3cf & -\frac{6}{7}e^2+2df \end{pmatrix}$$

and symplectic form

$$\Omega = \begin{pmatrix} 0 & -3a^2 & -c & -6b & 9a & 0 & -3 \\ 3a^2+c & 0 & -5a & 0 & -5 & 0 & \\ 6b & 5a & 0 & -15 & 0 & 0 & \\ -9a & 0 & 15 & 0 & 0 & 0 & \\ 0 & 5 & 0 & 0 & 0 & 0 & \\ 3 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

Each of the ideals $\mathrm{Pf}_{2\ell}$ is Poisson-closed, and defines a Cohen-Macaulay variety of codimension $3 - \ell + 1$.

3. For A_8, each of the ideals $\mathrm{Pf}_{2\ell}$ is Poisson-closed, and defines a Cohen-Macaulay variety of codimension $4 - \ell + 1$.

7.1 Case E_6

A versal deformation of E_6 is given by

$$F(x, y, a, b, c, d, e, f) = x^3 + y^4 + axy^2 + bxy + cy^2 + dx + ey + f.$$

We take the basis of $\Theta_S(-\log D)$ with symmetric Saito matrix χ equal to

$$\left(\begin{matrix}
2a & 5b & 6c & 8d \\
5b & \frac{-a^4}{6} - 4ac + 8d & \frac{a^2b}{2} + 9e & -\frac{a^3b}{12} - \frac{3bc+ae}{2} \\
6c & \frac{a^2b}{2} + 9e & -\frac{5b^2+2a^2c+10ad}{3} & +\frac{7ab^2}{12} - \frac{4a^2d}{3} + 12f \\
8d & -\frac{a^3b}{12} - \frac{3bc+ae}{2} & \frac{7ab^2}{12} - \frac{4a^2d}{3} + 12f & -\frac{a^2b^2}{24} + 4cd - \frac{7be}{2} + 6af \\
9e & \frac{ab^2-a^3c}{6} + \frac{a^2d-9c^2}{3} + 12f & \frac{7abc}{6} - \frac{13bd+4a^2e}{3} & \frac{5b^3-a^2bc}{12} - \frac{7abd}{6} - \frac{3ce}{2} \\
12f & \frac{abd}{6} - \frac{a^3e}{12} - \frac{3ce}{2} & -\frac{8d^2}{3} + \frac{7abe}{12} - 2a^2f & \frac{10b^2d-a^2be}{24} - \frac{4ad^2}{3} - \frac{9e^2}{4} + 6cf
\end{matrix}\right.$$

$$\left.\begin{matrix}
9e & 12f \\
\frac{ab^2-a^3c}{6} + \frac{a^2d-9c^2}{3} + 12f & \frac{abd}{6} - \frac{a^3e}{12} - \frac{3ce}{2} \\
\frac{7abc}{6} - \frac{13bd}{3} - \frac{4a^2}{3}e & -\frac{8d^2}{3} + \frac{7abe}{12} - 2a^2f \\
\frac{5b^3-a^2bc}{12} - \frac{7abd}{6} - \frac{3ce}{2} & \frac{10b^2d-a^2be}{24} - \frac{4ad^2}{3} - \frac{9e^2}{4} + 6cf \\
\frac{4b^2c}{3} - \frac{a^2c^2}{6} + \frac{8acd-8d^2-5abe-6a^2f}{3} & \frac{bcd}{2} + \frac{5b^2e-a^2ce}{12} + \frac{5ade}{6} - 3abf \\
\frac{bcd}{2} + \frac{5b^2e-a^2ce}{12} + \frac{5ade}{6} - 3abf & -\frac{4cd^2}{3} + \frac{11bde}{6} - \frac{a^2e^2}{24} - b^2f - 2adf
\end{matrix}\right) \tag{7.4}$$

The symplectic form Ω has matrix

$$\begin{pmatrix}
0 & -\frac{1}{15}ab & \frac{1}{5}c & \frac{2}{15}a^2 & 0 & \frac{1}{5} \\
\frac{1}{15}ab & 0 & 0 & 0 & \frac{1}{2} & 0 \\
-\frac{1}{5}c & 0 & 0 & 1 & 0 & 0 \\
-\frac{2}{15}a^2 & 0 & -1 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{5} & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{7.5}$$

The ideal I of entries in $\chi\Omega\chi$ is Cohen-Macaulay of codimension 3, and Poisson-closed. The ideal J of 4×4 Pfaffians is also Poisson closed, and has codimension 2 but projective dimension 3.

5. The following table shows the betti numbers (to be read from left to right) of minimal free resolutions of the ideals of Pfaffians, $\mathrm{Pf}_{2\ell}$, of the matrix $\chi\Omega\chi$ for singularities of type A_{2k} for $1 \leq k \leq 4$.

ℓ	A_2	A_4	A_6	A_8
1	1	3, 2	6, 8, 3	10, 20, 15, 4
2	–	1	5, 4	15, 24, 10
3	–	–	1	7, 6
4	–	–	–	1

(7.6)